Due Fri

1.5 – Elementary Matrices and a Method for Finding A^{-1}

Definition 1: Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

useful for developing theory

Definition 2: A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 1.5.1 Row Operations by Matrix Multiplication
If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

6. An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

a. $\boxed{E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

a) $R_1 \to -6R_1$ $\quad EA = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}$$

b) $R_2 \to R_2 - 4R_1$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$$

$$A: R_2 \to R_2 - 4R_1 = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$$

Pf Case I: E results from multiplying the i^{th} row of I by k.

Let $\vec{e}_1 = [1\ 0\ 0 \cdots 0]$, $\vec{e}_2 = [0\ 1\ 0 \cdots 0]$, $\vec{e}_m = [0\ 0 \cdots 1]$ and $k\vec{e}_i [0\ 0 \cdots k \cdots 0]$

$$EA = \begin{bmatrix} \vec{e}_1 A \\ \vec{e}_2 A \\ \vdots \\ k\vec{e}_i A \\ \vdots \\ e_m A \end{bmatrix},$$ which results in multiplying the i^{th} row of A by k.

7. Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}, F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

a. $EA = B$

b. $EB = A$

c. $EA = C$ $\quad R_3 \to R_3 - 2R_1$ $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d. $EC = A$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Check: $EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix} = C$ ✓

Theorem 1.5.2 Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Pf: Let E be the elementary matrix that results from performing a single elementary row operation on I and let E' be the matrix that results from performing the inverse operation on I. Then $EE' = I \Rightarrow E' = E^{-1}$.

Theorem 1.5.3 Equivalent Statements

iff

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all are true or all false.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has $\boxed{\text{only}}$ the trivial solution. — one and not more than one

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

$a \Leftrightarrow b$

$b \Leftrightarrow c$

$c \Leftrightarrow d$

Pf $a \Rightarrow b$: A invertible $\Rightarrow A^{-1}$ exists.

$$A\vec{x} = \vec{0} \Rightarrow A^{-1}\vec{x} = A^{-1}\vec{0}$$

$$\Rightarrow \vec{x} = \vec{0}$$

Suppose $\vec{x}_1$ is another solution.

Then $A\vec{x}_1 = \vec{0}$. $\Rightarrow A\vec{x} - A\vec{x}_1 = \vec{0} - \vec{0}$

$$= \vec{0}$$

$$A^{-1}A(\vec{x} - \vec{x}_1) = A^{-1}\vec{0} \Rightarrow \vec{x} - \vec{x}_1 = \vec{0}$$

$$\Rightarrow \vec{x} = \vec{x}_1.$$

b ⇒ c $A\vec{x} = \vec{0}$ has only the trivial solution, so the system contains no free variables

By Thm 1.2.1 rref has n nonzero rows.

By Thm 1.4.3, rref is I_n. ✓

c ⇒ d rref of A is I_n, so I_n can be obtained from A by a series of elementary row operations. We can express

$$I_n = E_r \cdots E_2 E_1 A \text{ by Thm 1.5.1.}$$

Then $A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} I_n$ $\quad (AB)^{-1} = B^{-1}A^{-1}$

So A is expressible as a product of elementary matrices

d ⇒ a From c ⇒ d, $(E_r \cdots E_2 E_1) A = I_n$,

so $E_r \cdots E_2 E_1 = A^{-1}$. A is invertible.

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

15. $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} = A$ $\quad [A \mid I_n] \rightarrow [I_n \mid A^{-1}]$

The inversion algorithm is based on $c \Rightarrow d$ in the proof above. Since $I_n = \underbrace{E_r \cdots E_2 E_1}\, A$, we have $\underbrace{A^{-1}} = E_r \cdots E_2 E_1$. Thus the elementary row operations that reduce A to I_n will yield A^{-1} when applied to I_n.

$$\begin{array}{cccc} & 4 & 3 & \\ & & & \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & \boxed{6} & \boxed{6} & 1 & 0 & 0 \\ \boxed{2} & 7 & \boxed{6} & 0 & 1 & 0 \\ \boxed{2} & \boxed{7} & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{ccc} 1 & 2 & \text{Last} \end{array}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{array}{rrrrrr} 2 & 7 & 6 & 0 & 1 & 0 \\ -2 & -6 & -6 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & -1 & 1 & 0 \end{array}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{array}{rrrrrr} 2 & 7 & 7 & 0 & 0 & 1 \\ -2 & -6 & -6 & -1 & 0 & 0 \\ \hline 0 & 1 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ \boxed{0} & 1 & 0 & -1 & 1 & 0 \\ \boxed{0} & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{array}{rrrrrr} 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & \boxed{0} & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 - 6R_3$

$$\begin{array}{rrrrrr} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & 6 & -6 \\ \hline 2 & 6 & 0 & 1 & 6 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \to R_1 - 6R_2$$

$$\begin{array}{cccccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & -6 & 0 & 6 & -6 & 0 \\ \hline 2 & 0 & 0 & 7 & 0 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \to \frac{1}{2}R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

6 5 4

18. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

1 2 3

last

Check:

$$A^{-1} = \begin{bmatrix} -4/5 & 3/5 & 1/5 & 1/5 \\ 3/2 & 0 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 4/5 & 2/5 & -1/5 & -1/5 \end{bmatrix}$$

19. Find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, k are all nonzero.

a. $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$ b. $\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

a. $\begin{bmatrix} 1/k_1 & 0 & 0 & 0 \\ 0 & 1/k_2 & 0 & 0 \\ 0 & 0 & 1/k_3 & 0 \\ 0 & 0 & 0 & 1/k_4 \end{bmatrix}$.

b. $\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$R_1 \to R_1 - R_2$

$\begin{array}{cccccccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline k & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{array}$

$R_3 \to R_3 - R_4$

$0 \; 0 \; k \; 0 \; 0 \; 0 \; 1 \; -1$

$\begin{bmatrix} 1/k & -1/k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/k & -1/k \\ 0 & 0 & 0 & 1 \end{bmatrix}$

27. Show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A, and then use that result to find a matrix C such that $CA = B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Corresponding elementary matrix

To obtain B from A:

$R_2 \to R_2 - R_1$

$$\begin{array}{ccc} 1 & 4 & 1 \\ -1 & -2 & -3 \\ \hline 0 & 2 & -2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -2 & 2 \\ \hline 1 & 0 & 5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{array}{ccc} 2 & 1 & 9 \\ -1 & 0 & -5 \\ \hline 1 & 1 & 4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

"B

$$\underbrace{(E_3 E_2 E_1)}_{C} A = B$$

$$C = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}}$$

24. Express the matrix and its inverse as products of elementary matrices.

$$A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

Basic idea: $\underbrace{(E_r \cdots E_2 E_1)}_{A^{-1}} A = I$

$\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_r^{-1}$

Reduce A to I_2, keeping track of the elementary row operations.

$$\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} \quad R_2 \to R_2 + 5R_1$$

$$\begin{array}{cc} -5 & 2 \\ 5 & 0 \\ \hline 0 & 2 \end{array}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad R_2 \to \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\boxed{A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}}$$

$$\boxed{A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

check

33. Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

Given: B is obtained from A by performing elementary row operations.

Let $E_1, E_2, \ldots, E_r$ be elementary matrices such that when the associated elementary row operations are applied to A, the result is B.

Then $E_r \cdots E_2 E_1 A = B$.

$m \times m \quad m \times n \qquad m \times n$

Since each E_i is invertible, (Thm 1.5.2)

$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} B.$$

$m \times m \quad m \times n$

Since E_i^{-1} is an elementary matrix, this shows that we can obtain A from B by a sequence of elementary row operations.